Some games and their topological consequences

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ICMC-USP (Partially supported by FAPESP)

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- and so on, for every $n \in \omega$.

At the end, BOB is declared the winner if $\bigcup_{n \in \omega} C_n = X$ and ALICE is declared the winner otherwise.

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Definition

We say that a topological space is a **Rothberger space** if ALICE does not have a winning strategy.

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First, it is good to note the following: the Rothberger game is only interesting for the Lindelöf case. In this way, we can suppose that every covering played by ALICE is countable.

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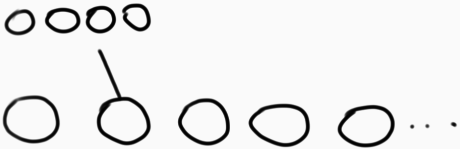

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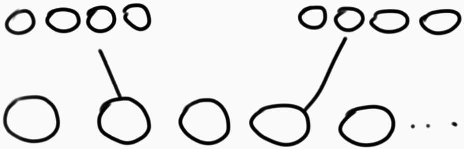

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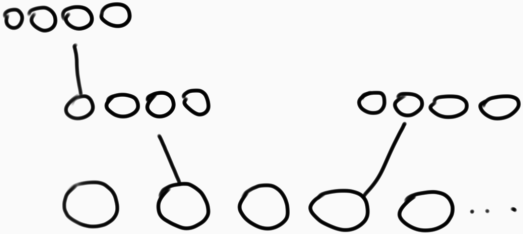

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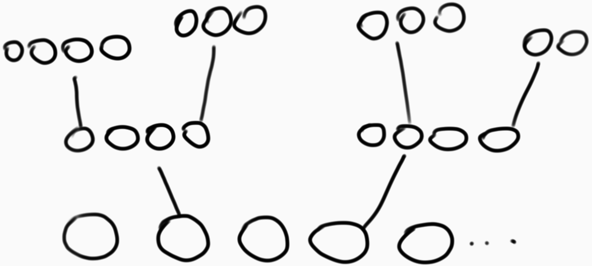

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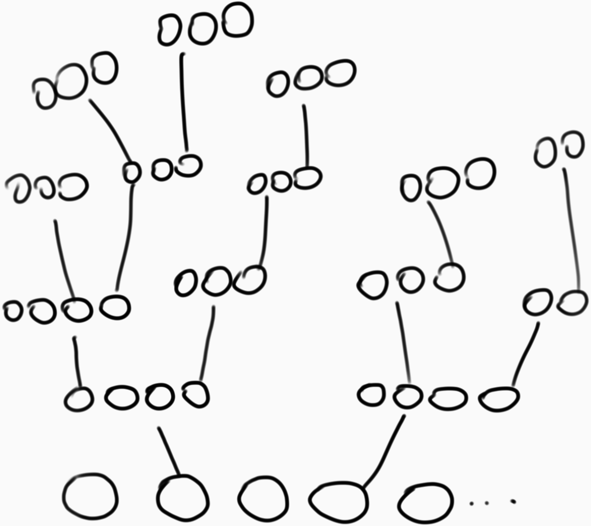

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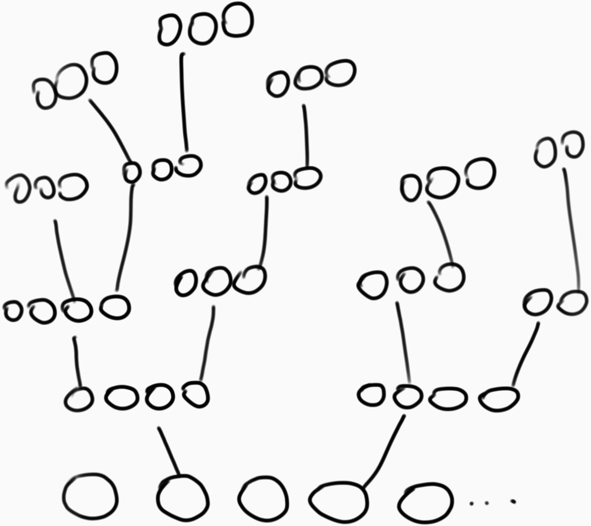

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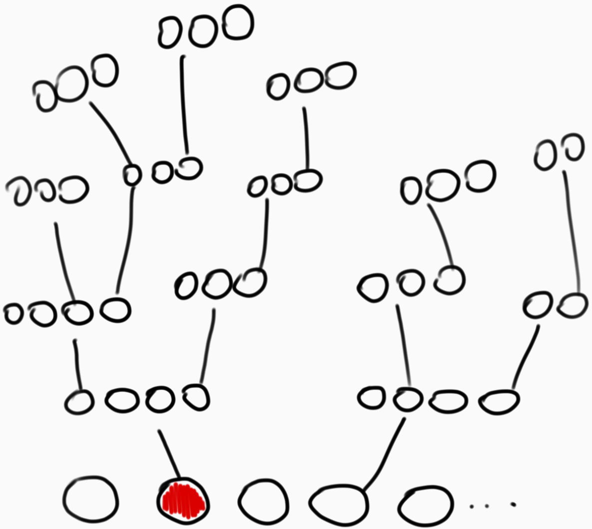

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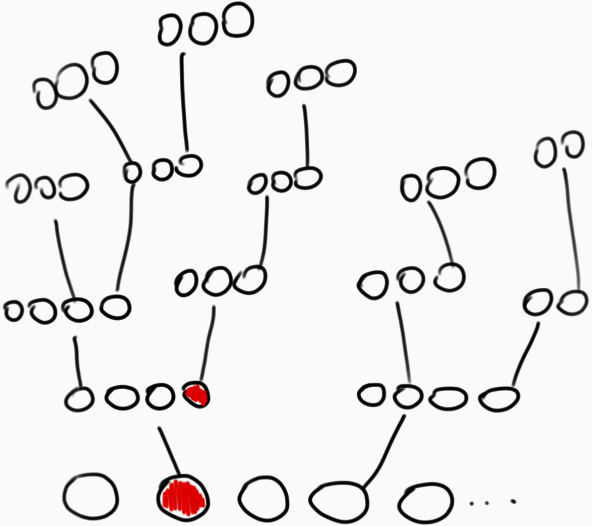

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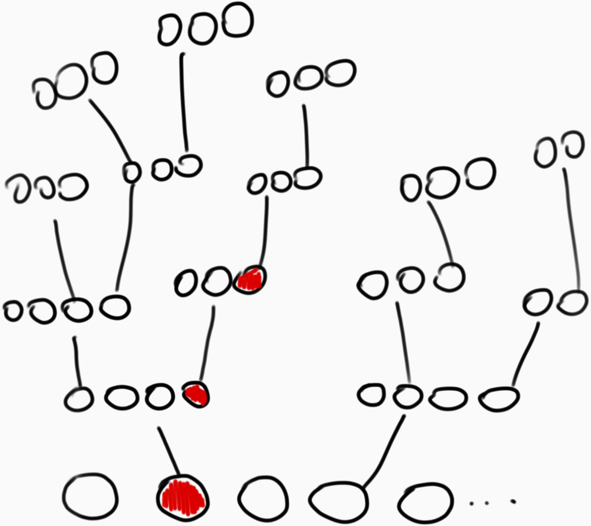

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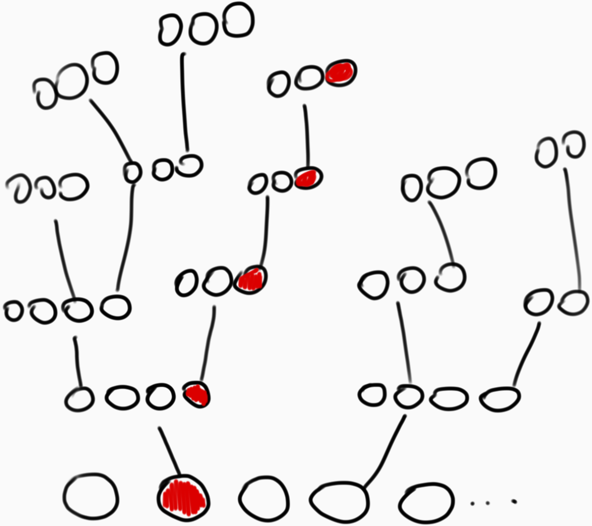

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- to change the length of the game (e.g., to $\omega + \omega$ innings or ω_1 innings);
- to put some kind of amnesia on the players (e.g. they only remember the previous inning, or just the number of the current inning).

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Looking at this proof, you can prove that compact Rothberger spaces are exactly the compacts that remain compact after any forcing extension.

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If ALICE does not have a winning strategy, the space is said to be a **Menger space**.

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- If the space is σ-compact, then Bob has a winning strategy.

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Theorem ([4, 2])

Every regular second countable space where BOB *has a winning strategy is σ-compact.*

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Let K_0 be the "intersection" of all these possible answers, as in the Mysterious Lemma. As before, K_0 is compact.

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Since K_0 is the intersection of countably many answers, let us fix countably many questions for Alice,

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Let us focus on the first inning.

First, fix a countable base $\mathcal{B}$ and suppose that ALICE will only play coverings made by basic open sets.

Even with this restriction, ALICE can ask uncountably many questions. But the important thing is that BOB has only countably many possible answers.

Let K_0 be the "intersection" of all these possible answers, as in the Mysterious Lemma. As before, K_0 is compact.

Since K_0 is the intersection of countably many answers, let us fix countably many questions for ALICE, i.e:

$$K_0 = \bigcap_{n \in \omega} \overline{\bigcup \sigma(\mathcal{C}_{\langle n \rangle})}$$

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Doing like this, we obtain $\langle K_s : s \in \omega^{<w} \rangle$.

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Suppose not.

Just go where x is not

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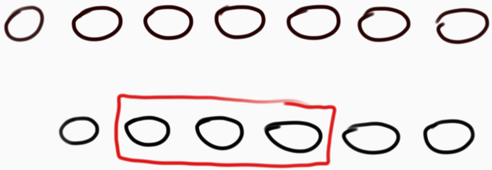

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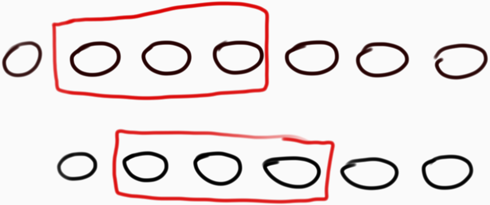

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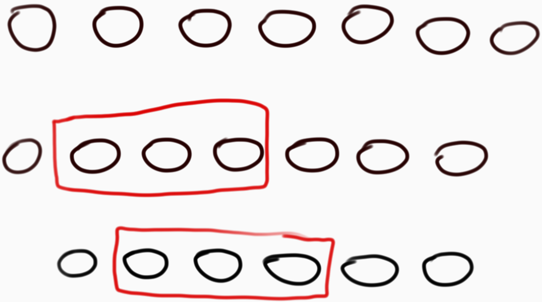

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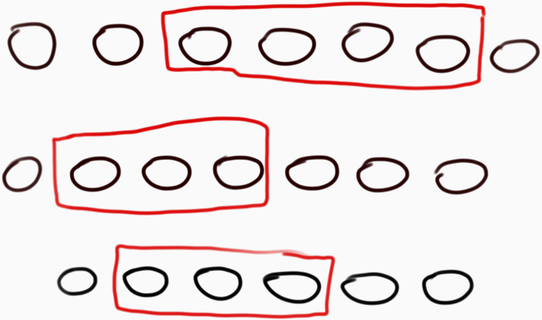

Another game

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- and so on, for every $n \in \omega$.

At the end, Alice is declared the winner if $\bigcup_{n\in\omega} V_n = X$ and Bob is declared the winner otherwise.

Relations between point-open and Rothberger

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The Rothberger game and the point-open game are dual

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Theorem ([1])

The Rothberger game and the point-open game are dual (i.e. Alice *has a winning strategy in one of the games iff* Bob *has a winning strategy in the other one)*

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Suppose that Alice has a winning strategy σ for the point-open game. Let us define how Bob should play in the Rothberger game.

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Alice plays an open covering $\mathcal{C}_0$. $\sigma(\langle\rangle) = x_0$ would be the first move of σ in the point-open game. Since $\mathcal{C}_0$ is a covering, there is a $C_0 \in \mathcal{C}_0$ such that $x_0 \in C_0$.

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Then ALICE plays $\mathcal{C}_1$. Let $x_1 = \sigma(\langle C_1 \rangle)$. As before, BOB can select $C_1 \in \mathcal{C}_1$ such that $x_1 \in C_1$. Then ALICE plays $\mathcal{C}_2$ and we fix $x_2 = \sigma(\langle C_1, C_2 \rangle)$ and so on.

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The point is, the sequence $\langle x_0, C_0, x_1, C_1, ... \rangle$ is a play of the point-open game where the strategy σ was used.

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Suppose that Alice has a winning strategy σ for the point-open game. Let us define how Bob should play in the Rothberger game.

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Then Alice plays $\mathcal{C}_1$. Let $x_1 = \sigma(\langle C_1 \rangle)$. As before, Bob can select $C_1 \in \mathcal{C}_1$ such that $x_1 \in C_1$. Then Alice plays $\mathcal{C}_2$ and we fix $x_2 = \sigma(\langle C_1, C_2 \rangle)$ and so on.

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Let us look at the first inning: Alice knows (from the other game) how to select elements from open coverings.

One nice implication

Now suppose that BOB has a winning strategy for the Rothberger game and we want to find a way for ALICE to win the point-open game.

Let us look at the first inning: ALICE knows (from the other game) how to select elements from open coverings. But she needs to begin this game playing a point.

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Lemma

Let σ be a strategy for Bob *in the Rothberger game.*

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Let σ be a strategy for Bob *in the Rothberger game. Then there is an $x \in X$ such that for every open set C such that $x \in C$, there is a $\mathcal{C}$ such that $C = \sigma(\mathcal{C})$.*

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Proof.

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Proof.

Suppose not. Then for every x there is a C_x such that $x \in C_x$ and C_x is not a possible answer from σ. Note that $\mathcal{C} = \{C_x : x \in X\}$ is an open covering. Since $\sigma(\mathcal{C}) \in \mathcal{C}$, we got a contradiction. $\square$

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